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Anelastic behaviour of materials under multiaxial strains

Part 1 *Theory*

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A theory of anelasticity is presented, where the relaxation of all the elastic compliances, as a function of orientation, is considered. The theory is developed for cubic and hexagonal crystals. It is shown that, in addition to the usual relaxations of Young's and shear moduli, anelastic phenomena related to the relaxation of Poisson's should be considered, and, an energetic interpretation of the phase lags obtained is given. Finally, some point defect symmetries and particular orientations are considered, as special situations of the general formalism presented.

1. Introduction

The anelastic behaviour of materials is usually well described by the standard anelastic solid model [1]. The anelastic properties of such a solid can be expressed by the linear differential equation

$$\varepsilon + \tau_\sigma \dot{\varepsilon} = J_r \sigma + J_u \tau_\sigma \dot{\sigma} \tag{1}$$

where σ is the applied stress, ε the strain, J_r and J_u are the relaxed and unrelaxed compliances, respectively, and τ_σ is the relaxation time at constant stress. The dot indicates derivative with respect to the time, t.

For a periodic applied stress, expressed by

$$\sigma = \sigma_0 \exp(i\omega t) \tag{2}$$

and the response strain

$$\varepsilon = (\varepsilon^{(1)} - i\varepsilon^{(2)}) \exp(i\omega t) \tag{3}$$

Equation 1 yields the complex compliance

$$J(\omega) = J^{(1)}(\omega) - iJ^{(2)}(\omega) = \frac{\varepsilon}{\sigma} \tag{4}$$

where

$$J^{(1)}(\omega) = J_u + \frac{\delta J}{1 + \omega^2 \tau_\sigma^2} \tag{5a}$$

and

$$J^{(2)}(\omega) = \delta J \frac{\omega \tau_\sigma}{1 + \omega^2 \tau_\sigma^2} \tag{5b}$$

Equations 5a and b are the so called Debye equations, with

$$\delta J = J_r - J_u \tag{6}$$

Moreover, If the phase lag between the stress and the strain is defined by

$$\tan \phi(\omega) = \frac{\varepsilon^{(2)}(\omega)}{\varepsilon^{(1)}(\omega)} = \frac{J^{(2)}(\omega)}{J^{(1)}(\omega)} \tag{7}$$

it can be shown that

$$\tan \phi(\omega) = \frac{1}{2\pi} \frac{\Delta W}{W} \tag{8}$$

where ΔW and W are the lost and stored energies per cycle of vibration, respectively. Furthermore

$$\tan \phi(\omega) = \frac{\Delta_M}{(1 + \Delta_M)^{1/2}} \frac{\omega \bar{\tau}}{1 + \omega^2 \bar{\tau}^2} \tag{9}$$

where

$$\Delta_M = \frac{\delta J}{J_u} \tag{10}$$

and

$$\bar{\tau} = \frac{\tau_\sigma}{(1 + \Delta_M)^{1/2}} \tag{11}$$

Equations 5b and 9 lead to Debye peaks when plotted as a function of $\omega\tau$.

The relationships described are generally used to represent the anelastic behaviour of specimens excited under simple situations, like longitudinally or in torsion. In these situations, only the relaxations of Young's or shear moduli are measured. Furthermore, for single crystals only the orientation dependence of the relaxation of these two moduli are generally obtained [1].

The more complicated case of multiaxial stresses have been considered, from an engineering point of view, both by Lazan [2] and by Alfrey and Gurnee [3]. Some introductory considerations have been made by Wert [4] to the case of multiaxial strains.

It is the purpose of this paper to extend the formalism to all the elastic compliances, as a function of orientation, both for cubic and hexagonal symmetries. This will allow the determination of the relaxed and unrelaxed Poisson's ratios, in two orthogonal directions located in the plane perpendicular to the direction of the applied stress. With this information, it is possible to study the relaxation behaviour in directions perpendicular to those corresponding to the excitation, and, an energetic interpretation will be given to the phase lag obtained. Finally, some point defect symmetries and particular orientations will be considered, as special situations of the general formalism presented in the paper.

2. Theory

Generalized Hooke's law can be expressed, in terms of the commonly used single index notation, as [5]

$$\sigma_i = \sum_{j=1}^{6} c_{ij}\varepsilon_j \qquad i, j = 1, \ldots, 6 \tag{12}$$

where c_{ij} are the elastic stiffness constants. In terms of the elastic compliances

$$\varepsilon_i = \sum_{j=1}^{6} s_{ij}\sigma_j \tag{13}$$

The number of different elastic stiffness or elastic compliances stays between a maximum of 21 and a minimum of 2 for isotropic solids. Further simplifications of Hooke's law for crystals can be made if, instead of the usual components of stress and strain, six independent linear combinations of these are chosen, which poses certain fundamental symmetry properties associated with the crystal in question. These linear combinations, which are known as the symmetry coordinates of stress and strain, or as symmetrized stresses and strains, are obtained by means of group theory [6]. The symmetrized coordinates are listed, for cubic and hexagonal crystals, in Tables I and II, respectively, and they are classified as Type I and Type II.

The special feature of strains of Type I is that a crystal subjected to such a strain is not lowered in symmetry by the deformation. On the other hand, a crystal under a Type II strain is lowered in symmetry. Furthermore, whenever a symmetrized stress is decoupled from all the symmetrized strains, except the one which corresponds to it, Hooke's law reverts to the simple form

$$\varepsilon_\gamma = S_\gamma \sigma_\gamma \tag{14}$$

where γ denotes the symmetry designation and S_γ is the appropriate symmetrized compliance. In the hexagonal Type I symmetrized coordinates, however, complete decoupling does not occur and the situation is more complex. For lower symmetry crystals decoupling occurs less frequently until finally, for the triclinic case, all six stresses and strain components are of Type I and a set of completely coupled equations is obtained. The reason is that triclinic crystals show no symmetry and there is no simplification of Hooke's law as a consequence of symmetry considerations.

In order to generalize the equations of elasticity of crystals to allow for time-dependent effects, the validity of the standard anelastic solid model will be accepted, for each symmetrized coordinate decoupled one from another. In this case

$$\varepsilon_\gamma + \tau_{\sigma\gamma}\dot{\varepsilon}_\gamma = S_{\gamma r}\sigma_\gamma + \tau_{\sigma\gamma}S_{\gamma u}\dot{\sigma}_\gamma \tag{15}$$

where r and u denote relaxed and unrelaxed compliances, respectively. Furthermore, in the theory of point defects relaxation, when defects of only a single specie are present, only compliances of Type II may undergo relaxation [1]. Such a situation will be assumed in the theoretical development that will follow.

TABLE I Symmetrical stresses, strains and compliances of Type I

Crystal system	Stress	Compliance	Strain
Cubic	$\sigma_1 + \sigma_2 + \sigma_3 \rightarrow$	$s_{11} + 2s_{12} \rightarrow$	$\varepsilon_1 + \varepsilon_2 + \varepsilon_3$
Hexagonal	$\dfrac{\sigma_1 + \sigma_2}{\sqrt{2}} \rightarrow$	$s_{11} + s_{12} \rightarrow$	$\dfrac{\varepsilon_1 + \varepsilon_2}{\sqrt{2}}$
	$\searrow \nearrow$	$\sqrt{2}s_{13}$	$\searrow \nearrow$
	$\sigma_3 \rightarrow$	$s_{33} \rightarrow$	ε_3

2.1. Cubic symmetry. Longitudinal stress

For a cubic crystal under a uniaxial stress applied along X_1', Fig. 1, in the two index notation

$$\sigma_{ij} = a^+_{ik}a^+_{jl}\sigma'_{kl} \tag{16}$$

where $\{a^+\}$ is the inverse matrix of the orthogonal transformation $X_i \rightarrow X_i'$. Moreover, from Tables I and II

$$\begin{aligned}
\varepsilon_1 + \varepsilon_2 + \varepsilon_3 &= (S_{11} + 2S_{12})(\sigma_1 + \sigma_2 + \sigma_3)\\
2\varepsilon_1 - \varepsilon_2 - \varepsilon_3 &= (S_{11} - S_{12})(2\sigma_1 - \sigma_2 - \sigma_3)\\
\varepsilon_2 - \varepsilon_3 &= (S_{11} - S_{12})(\sigma_2 - \sigma_3)\\
\varepsilon_4 &= S_{44}\sigma_4\\
\varepsilon_5 &= S_{44}\sigma_5\\
\varepsilon_6 &= S_{44}\sigma_6
\end{aligned} \tag{17}$$

and, on combining Equations 16 and 17 leads to

$$\begin{aligned}
\varepsilon_1 + \varepsilon_2 + \varepsilon_3 &= (S_{11} + 2S_{12})(a^{+2}_{11} + a^{+2}_{21} + a^{+2}_{31})\sigma_1'\\
2\varepsilon_1 - \varepsilon_2 - \varepsilon_3 &= (S_{11} - S_{12})(2a^{+2}_{11} - a^{+2}_{21} - a^{+2}_{31})\sigma_1'\\
\varepsilon_2 - \varepsilon_3 &= (S_{11} - S_{12})(a^{+2}_{21} - a^{+2}_{31})\sigma_1'\\
\varepsilon_4 &= S_{44}a^+_{31}a^+_{21}\sigma_1'\\
\varepsilon_5 &= S_{44}a^+_{31}a^+_{11}\sigma_1'\\
\varepsilon_6 &= S_{44}a^+_{21}a^+_{11}\sigma_1'
\end{aligned} \tag{18}$$

TABLE II Symmetrical stresses, strains and compliances of Type II

Crystal system	Stress	Compliance	Strain
Cubic	$2\sigma_1 - \sigma_2 - \sigma_3 \rightarrow$	$s_{11} - s_{12} \rightarrow$	$2\varepsilon_1 - \varepsilon_2 - \varepsilon_3$
	$\sigma_2 - \sigma_3 \rightarrow$	$s_{11} - s_{12} \rightarrow$	$\varepsilon_2 - \varepsilon_3$
	$\sigma_4 \rightarrow$	$s_{44} \rightarrow$	ε_4
	$\sigma_5 \rightarrow$	$s_{44} \rightarrow$	ε_5
	$\sigma_6 \rightarrow$	$s_{44} \rightarrow$	ε_6
Hexagonal	$\sigma_4 \rightarrow$	$s_{44} \rightarrow$	ε_4
	$\sigma_5 \rightarrow$	$s_{44} \rightarrow$	ε_5
	$\sigma_1 - \sigma_2 \rightarrow$	$s_{11} - s_{12} \rightarrow$	$\varepsilon_1 - \varepsilon_2$
	$\sigma_6 \rightarrow$	$s_{11} - s_{12} \rightarrow$	$\varepsilon_6/2$

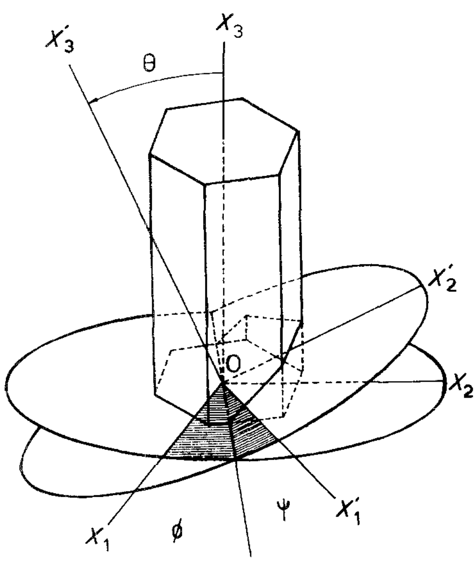

Figure 1 Cubic cell and Euler's angles.

The time dependent generalization of these equations is given by

$$
\begin{aligned}
&(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \tau_{\sigma(S_{11}+2S_{12})}(\dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3) \\
&\quad = (S_{11} + 2S_{12})\sigma_1' + [(S_{11} + 2S_{12}) \\
&\qquad - \delta_{(S_{11}+2S_{12})}]\tau_{\sigma(S_{11}+2S_{12})}\dot{\sigma}_1' \\
&(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) + \tau_{\sigma(S_{11}-S_{12})}(2\dot{\varepsilon}_1 - \dot{\varepsilon}_2 - \dot{\varepsilon}_3) \\
&\quad = (2a_{11}^{+2} - a_{21}^{+2} - a_{31}^{+2})\{(S_{11} - S_{12})\sigma_1' \\
&\qquad + [(S_{11} - S_{12}) - \delta_{(S_{11}-S_{12})}] \\
&\qquad \times \tau_{\sigma(S_{11}-S_{12})}\dot{\sigma}_1'\} \\
&(\varepsilon_2 - \varepsilon_3) + \tau_{\sigma(S_{11}-S_{12})}(\dot{\varepsilon}_2 - \dot{\varepsilon}_3) \\
&\quad = (a_{21}^{+2} - a_{31}^{+2})\{(S_{11} - S_{12})\sigma_1' \\
&\qquad + [(S_{11} - S_{12}) - \delta_{(S_{11}-S_{12})}] \\
&\qquad \times \tau_{\sigma(S_{11}-S_{12})}\dot{\sigma}_1'\} \\
&\varepsilon_4 + \tau_{\sigma_{S_{44}}}\dot{\varepsilon}_4 = a_{31}^{+}a_{21}^{+}[S_{44}\sigma_1' + (S_{44} - \delta_{S_{44}})\tau_{\sigma_{44}}\dot{\sigma}_1'] \\
&\varepsilon_5 + \tau_{\sigma_{S_{44}}}\dot{\varepsilon}_5 = a_{31}^{+}a_{11}^{+}[S_{44}\sigma_1' + (S_{44} - \delta_{S_{44}})\tau_{\sigma_{S_{44}}}\dot{\sigma}_1'] \\
&\varepsilon_6 + \tau_{\sigma_{S_{44}}}\dot{\varepsilon}_6 = a_{21}^{+}a_{11}^{+}[S_{44}\sigma_1' + (S_{44} - \delta_{S_{44}})\tau_{\sigma_{44}}\dot{\sigma}_1']
\end{aligned}
\tag{19}
$$

where the subindex denotes the respective symmetrized coordinates for τ_σ and δ.

For a time dependent sinusoidal stress and an analog response for the strain, that is,

$$\sigma_1' = \sigma_{01} \exp(i\omega t)$$

$$\varepsilon_j = (e_j^{(1)} - \varepsilon_j^{(2)}) \exp(i\omega t) \qquad j = 1, \ldots 6 \tag{20}$$

and taking into account that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ is a Type I strain so that $\tau_{\sigma_{(s_{11}+2s_{12})}} = 0$ and $\delta_{(s_{11}+2s_{12})} = 0$, after a long algebraic and analytical treatment it can be shown that

$$
\begin{aligned}
\varepsilon_1 = &\{(s_{11} - s_{12})(3a_{11}^{+2} - 1) + (s_{11} + 2s_{12}) \\
&+ \{[(s_{11} - s_{12}) - \delta_{(s_{11}-s_{12})}](3a_{11}^{+2} - 1) \\
&+ (s_{11} + 2s_{12})\}\,\omega^2\tau^2_{\sigma(s_{11}-s_{12})} \\
&- i\omega\tau_{\sigma(s_{11}-s_{12})}\delta_{(s_{11}-s_{12})}(3a_{11}^{+2} - 1)\}\,\sigma_1'/ \\
&[3(1 + \omega^2\tau^2_{\sigma(s_{11}-s_{12})})]
\end{aligned}
\tag{21}
$$

$$\varepsilon_4 = \frac{s_{44} + (s_{44} - \delta_{s_{44}})\omega^2\tau\sigma^2_{S_{44}} - i\omega\tau_{0_{s_{44}}}\delta_{s_{44}}}{(1 + \omega^2\tau^2_{\sigma_{s_{44}}})}\, a_{31}^{+}a_{21}^{+}\sigma_1'$$

Analogous expressions can be obtained for the other strains. Of interest for this paper are the strains ε_j' which in the two index notation can be expressed as

$$\varepsilon_{jk}' = a_{jl}a_{km}\varepsilon_{lm} \tag{22}$$

Then, on combining Equations 21 and 22 leads to

$$
\begin{aligned}
\varepsilon_1' = &|\{(s_{11} + 2s_{12}) + 2(s_{11} - s_{12})(1 - 3\Gamma_{11}) \\
&+ \omega^2\tau^2_{\sigma(s_{11}-s_{12})}\{(s_{11} + 2s_{12}) \\
&+ 2[(s_{11} - s_{12}) - \delta_{(s_{11}-s_{12})}](1 - 3\Gamma_{11})\} \\
&- i2\omega\tau_{\sigma(s_{11}-s_{12})}\delta_{(s_{11}-s_{12})}(1 - 3\Gamma_{11})\}/ \\
&[3(1 + \omega^2\tau^2_{\sigma(s_{11}-s_{12})})] + \Gamma_{11}[s_{44} + (s_{44} - \delta_{s_{44}})\omega^2\tau^2_{\sigma_{s_{44}}} \\
&- i\omega\delta_{s_{44}}\tau_{\sigma_{44}}]/(1 + \omega^2\tau^2_{\sigma_{s_{44}}})|\sigma_1'
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\varepsilon_2' = &|\{(s_{11} + 2s_{12}) - (s_{11} - s_{12})(1 - 3\Gamma_{12}) \\
&+ \omega^2\tau^2_{\sigma(s_{11}-s_{12})}\{(s_{11} + 2s_{12}) \\
&- [(s_{11} - s_{12}) - \delta_{(s_{11}-s_{12})}](1 - 3\Gamma_{12})\} \\
&+ i\omega\tau_{\sigma(s_{11}-s_{12})}\delta_{(s_{11}-s_{12})}(1 - 3\Gamma_{12})\}/ \\
&[3(1 + \omega^2\tau^2_{\sigma(s_{11}-s_{12})})] - \Gamma_{12}[s_{44} + (s_{44} - \delta_{s_{44}})\omega^2\tau^2_{\sigma_{s_{44}}} \\
&- i\omega\tau_{\sigma_{s_{44}}}\delta_{s_{44}}]/[2(1 + \omega^2\tau^2_{\sigma_{s_{44}}})]|\sigma_1'
\end{aligned}
$$

where

$$\Gamma_{11} = a_{11}^2a_{12}^2 + a_{11}^2a_{13}^2 + a_{12}^2a_{13}^2 \tag{24}$$

and

$$\Gamma_{12} = a_{11}^2a_{21}^2 + a_{12}^2a_{22}^2 + a_{13}^2a_{23}^2 \tag{25}$$

are orientation factors for the cubic symmetry. a_{ij} are the components of the matrix $\{a\}$ of the orthogonal transformation $X_i \to X_i'$. Equation 23 can also be written as

$$
\begin{aligned}
\varepsilon_1' = &\left[\frac{(s_{11} + 2s_{12}) + 2(s_{11} - s_{12})_u(1 - 3\Gamma_{11})}{3}\right. \\
&\left. + \Gamma_{11}s_{44u}\right] + \frac{\frac{2}{3}(1 - 3\Gamma_{11})\delta_{(s_{11}-s_{12})}}{(1 + \omega^2\tau^2_{\sigma(s_{11}-s_{12})})} + \frac{\Gamma_{11}\delta_{s_{44}}}{1 + \omega^2\tau^2_{\sigma_{s_{44}}}} \\
&- i\left[\frac{\frac{2}{3}(1 - 3\Gamma_{11})\delta_{(s_{11}-s_{12})}\omega\tau_{\sigma(s_{11}-s_{12})}}{1 + \omega^2\tau^2_{\sigma(s_{11}-s_{12})}} + \frac{\Gamma_{11}\omega\tau_{\sigma_{s_{44}}}\delta_{s_{44}}}{1 + \omega^2\tau^2_{\sigma_{s_{44}}}}\right] \\
\varepsilon_2' = &\left[\frac{(s_{11} + 2s_{12}) - (s_{11} - s_{12})_u(1 - 3\Gamma_{12})}{3}\right. \\
&\left. - \frac{\Gamma_{12}}{2}s_{44u}\right] + \frac{\frac{1}{3}(1 - 3\Gamma_{12})\delta_{(s_{11}-s_{12})}}{(1 + \omega^2\tau^2_{\sigma(s_{11}-s_{12})})} - \frac{\Gamma_{12}\delta_{s_{44}}/2}{1 + \omega^2\tau^2_{\sigma_{s_{44}}}} \\
&+ i\left[\frac{\omega\tau_{\sigma(s_{11}-s_{12})}(1 - 3\Gamma_{12})\delta_{(s_{11}-s_{12})}}{3(1 + \omega^2\tau^2_{\sigma(s_{11}-s_{12})})} - \frac{\Gamma_{12}\omega\tau_{\sigma_{s_{44}}}\delta_{s_{44}}}{2(1 + \omega^2\tau^2_{\sigma_{s_{44}}})}\right]
\end{aligned}
\tag{26}
$$

The expressions given by Equation 26 are formally

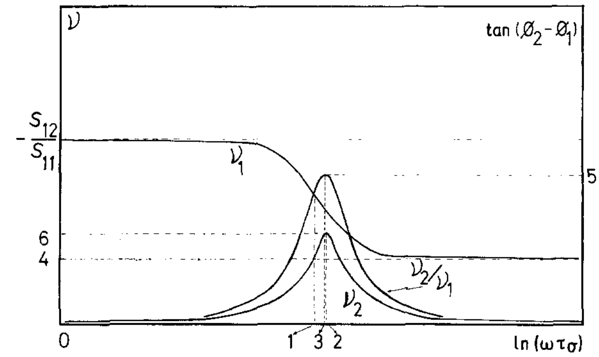

Figure 2 Complex Poisson's ratio and $\tan(\phi_2 - \phi_1)$ against $\ln \omega\tau_\sigma$. Some characteristic values, indicated from 1 to 6 are given in Table III.

similar to the equations of the five parameters model containing two Voigt units [1]. The subscript u has the same meaning as for Equation 1, that is, the unrelaxed values. The compliances without subscript, as in the previous equations, will indicate relaxed compliances. Furthermore,

$$J(\omega) = \left[J_u + \frac{\delta J^{(1)}}{1 + \omega^2\tau_\sigma^{(1)2}} + \frac{\delta J^{(2)}}{1 + \omega^2\tau_\sigma^{(2)2}}\right] - i\left[\delta J^{(1)} \frac{\omega\tau_\sigma^{(1)}}{1 + \omega^2\tau_\sigma^{(1)2}} + \delta J^{(2)} \frac{\omega\tau_\sigma^{(2)}}{1 + \omega^2\tau_\sigma^{(2)2}}\right]$$

In addition, if $\Gamma_{11} = \Gamma_{12} = 0$ then

$$\varepsilon_1' = \frac{s_{11} + [s_{11} - \frac{2}{3}\delta_{(s_{11}-s_{12})}]\omega^2\tau^2_{\sigma(s_{11}-s_{12})} - i\omega\tau_{\sigma(s_{11}-s_{12})}\frac{2}{3}\delta_{(s_{11}-s_{12})}}{1 + \omega^2\tau^2_{\sigma(s_{11}-s_{12})}}\sigma_1'$$

$$\varepsilon_2' = \frac{s_{12} + [s_{12} + \frac{1}{3}\delta_{(s_{11}-s_{12})}]\omega^2\tau^2_{\sigma(s_{11}-s_{12})} + i\omega\tau_{\sigma(s_{11}-s_{12})}\frac{1}{3}\delta_{(s_{11}-s_{12})}}{1 + \omega^2\tau^2_{\sigma(s_{11}-s_{12})}}\sigma_1' \quad (28)$$

and Poisson's ratio is given by

$$\nu_{12} = \nu^{(1)} - i\nu^{(2)} = -\varepsilon_2'/\varepsilon_1' \quad (29)$$

with

$$\nu^{(1)} = \frac{-s_{11}s_{12} + \left(s_{12} + \frac{\delta_{(s_{11}-s_{12})}}{3}\right)\left(s_{11} - \frac{2\delta_{(s_{11}-s_{12})}}{3}\right)\omega^2\tau^2_{\sigma(s_{11}-s_{12})}}{s_{11}^2 + (s_{11} - \frac{2}{3}\delta_{(s_{11}-s_{12})})^2\omega^2\tau^2_{\sigma(s_{11}-s_{12})}} \quad (30)$$

and

$$\nu^{(2)} = \frac{\omega\tau_{\sigma(s_{11}-s_{12})}\delta_{(s_{11}-s_{12})}(s_{11} + 2s_{12})}{s_{11}^2 + (s_{11} - \frac{2}{3}\delta_{(s_{11}-s_{12})})^2\omega^2\tau^2_{\sigma(s_{11}-s_{12})}} \quad (31)$$

Moreover, it is easy to see that if

$$\omega\tau_{\sigma(s_{11}-s_{12})} \to 0 \quad \nu^{(1)} \cong -\frac{s_{12}}{s_{11}}; \quad \nu^{(2)} = 0 \quad (32a)$$

$$\omega\tau_{\sigma(s_{11}-s_{12})} \to \infty \quad \nu^{(1)} \cong -\frac{s_{12} + \delta_{(s_{11}-s_{12})/3}}{s_{11} - \frac{2}{3}\delta_{(s_{11}-s_{12})}}; \quad \nu^{(2)} = 0 \quad (32b)$$

The relaxation curves described by Equations 30 and 31 are similar to those of Young's and shear moduli, in isotropic materials. In fact

$$\frac{\partial\nu^{(2)}}{\partial(\omega\tau_\sigma)} = 0 \Rightarrow \omega\tau_{\sigma_{s_{11}-s_{12}}} = \frac{s_{11}}{s_{11} - \frac{2}{3}\delta_{(s_{11}-s_{12})}} \quad (33)$$

and $\nu^{(2)}$ has a maximum at $\omega\tau_\sigma \simeq 1$, as for a Debye curve. In addition, since for

$$\frac{\partial^2\nu^{(1)}}{\partial(\omega\tau_\sigma)^2} = 0 \Rightarrow \omega\tau_{\sigma(s_{11}-s_{12})} = \frac{s_{11}}{\sqrt{3}\,(s_{11} - \frac{2}{3}\delta_{(s_{11}-s_{12})})} \quad (34)$$

the modulus curve shows an inflection point at $\omega\tau_\sigma = 1/\sqrt{3}$, slightly displaced with respect to the classical value for Young's and shear moduli, as shown in Fig. 2.

From another point of view, according to the theories of elasticity and for cubic symmetry [7]

$$\nu_{12}' = -\frac{s_{12}'}{s_{11}'} = -\frac{s_{12} + (s_{11} - s_{12} - \frac{1}{2}s_{44})\Gamma_{12}}{s_{11} - 2(s_{11} - s_{12} - \frac{1}{2}s_{44})\Gamma_{11}} \quad (35)$$

and on using the same procedure as in the case of errors theory for $\delta\nu_{12}' \ll \nu_{12}'$ it can be shown that

$$\frac{\delta\nu_{12}'}{\nu_{12}'} = -\left[\frac{\frac{1}{3}(\delta_{(s_{11}+2s_{12})} - \delta_{(s_{11}-s_{12})}) + (\delta_{(s_{11}-s_{12})} - \delta_{s_{44/2}})\Gamma_{12}}{s_{12} + (s_{11} - s_{12} - s_{44/2})\Gamma_{12}} - \frac{\frac{1}{3}(\delta_{(s_{11}+2s_{12})} + 2\delta_{(s_{11}-s_{12})}) - (2\delta_{(s_{11}-s_{12})} - \delta_{(s_{44})})\Gamma_{11}}{s_{11} - 2(s_{11} - s_{12} - s_{44/2})\Gamma_{11}}\right] \quad (36)$$

If $\Gamma_{12} = \Gamma_{11} = 0$ and $\delta_{(s_{11}+2s_{12})} = 0$, Equation 36 reduces to

$$\delta\nu_{12} = -\frac{\delta_{(s_{11}-s_{12})}}{3s_{11}}\left(1 + 2\frac{s_{12}}{s_{11}}\right) \quad (37)$$

Moreover, if $\delta_{(s_{11}-s_{12})} \ll (s_{11} - s_{12})$, Equation (32b) leads to

$$\nu_{12}^{(1)} \cong -\frac{s_{12}}{s_{11}} - \frac{\delta_{(s_{11}-s_{12})}}{3s_{11}}\left(1 + 2\frac{s_{12}}{s_{11}}\right) \quad (38)$$

and

$$\delta\nu_{12} \cong -\frac{\delta_{(s_{11}-s_{12})}}{3s_{11}}\left(1 + \frac{2s_{12}}{s_{11}}\right) \quad (39)$$

which coincides with Equation 37.

2.2. Cubic symmetry. Shear stress

If a shear stress σ_{23}' is applied to a cubic cell, a procedure similar to the one used for the previous case leads to

$$\varepsilon'_{23} = \left\{2\Gamma_{23}\frac{(s_{11}-s_{12})+[(s_{11}-s_{12})-\delta_{(s_{11}-s_{12})}]\,\omega^2\tau\sigma^2_{s_{11}-s_{12}}-i\delta_{(s_{11}-s_{12})}\,\omega\tau_{\sigma_{(s_{11}-s_{12})}}}{(1+\omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}})}\right.$$
$$\left.+\tfrac{1}{2}(1-2\Gamma_{23})\frac{s_{44}+(s_{44}-\delta_{s_{44}})\,\omega^2\tau^2_{\sigma_{s_{44}}}-i\omega\tau_{\sigma_{s_{44}}}\delta s_{44}}{(1+\omega^2\tau^2_{\sigma_{s_{44}}})}\right\}\sigma'_{23} \qquad (40)$$

For a specimen with cylindrical symmetry, an integration over ψ between 0 and 2π should be performed [8]. Then, if $\sigma'_{23} = 2\sigma'_4$

$$\varepsilon'_4 = \left\{4\Gamma_{33}\frac{(s_{11}-s_{12})+[(s_{11}-s_{12})-\delta_{(s_{11}-s_{12})}]\,\omega^2\tau\sigma^2_{s_{11}-s_{12}}-i\omega\tau^2_{\sigma_{s_{11}-s_{12}}}\delta_{(s_{11}-s_{12})}}{(1+\omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}})}\right.$$
$$\left.+(1-2\Gamma_{33})\frac{s_{44}+(s_{44}-\delta_{s_{44}})\,\omega^2\tau^2_{\sigma_{s_{44}}}-i\omega\tau_{\sigma_{s_{44}}}\delta_{s_{44}}}{(1+\omega^2\tau^2_{\sigma_{s_{44}}})}\right\}\sigma'_4 \qquad (41)$$

In the particular case where $\omega\tau_\sigma \to 0$

$$\varepsilon'_4 = [4(s_{11}-s_{12})\Gamma_{33} + (1-2\Gamma_{33})s_{44}]\,\sigma'_4 \qquad (42)$$

and if $\omega\tau_\sigma \to \infty$

$$\varepsilon'_4 = \{4[(s_{11}-s_{12})-\delta_{(s_{11}-s_{12})}]\Gamma_{33} + (1-2\Gamma_{33})(s_{44}-\delta_{s_{44}})\}\,\sigma'_4 \qquad (43)$$

Moreover, shear modulus is given by

$$G^{-1} = \frac{\varepsilon'_4}{\sigma'_4} \qquad (44)$$

and

$$\delta G^{-1} = G^{-1}_{\omega\tau_\sigma\to\infty} - G^{-1}_{\omega\tau_\sigma\to 0} = 4\Gamma_{33}\delta_{(s_{11}-s_{12})} + (1-2\Gamma_{33})\delta_{s_{44}} \qquad (45)$$

which is the expression commonly reported in the literature [1].

2.3. Hexagonal symmetry. Longitudinal stress

The treatment of hexagonal crystals under longitudinal sinusoidal stresses can be carried on in a similar way, starting from Tables I and II. The elastic relations are

$$\begin{aligned}
\frac{1}{\sqrt{2}}(\varepsilon_1+\varepsilon_2) &= \sqrt{2}\,s_{13}\sigma_3 + (s_{11}+s_{12})\frac{(\sigma_1+\sigma_2)}{\sqrt{2}}\\
\varepsilon_3 &= s_{33}\sigma_3 + s_{13}(\sigma_1+\sigma_2)\\
\varepsilon_1-\varepsilon_2 &= (s_{11}-s_{12})(\sigma_1-\sigma_2)\\
\varepsilon_4 &= s_{44}\sigma_4\\
\varepsilon_5 &= s_{44}\sigma_5\\
\varepsilon_6 &= 2(s_{11}-s_{12})\sigma_6
\end{aligned} \qquad (46)$$

If the stress is applied along X'_3 (see Fig. 3) the anelastic relationships associated with Equation 46 are given by

$$\frac{1}{\sqrt{2}}(\varepsilon_1+\varepsilon_2) = \left[\sqrt{2}s_{13}a^{+2}_{33} + \frac{(a^{+2}_{13}+a^{+2}_{23})(s_{11}+s_{12})}{\sqrt{2}}\right]\sigma'_3 \qquad (47a)$$

$$\varepsilon_3 = [a^{+2}_{33}s_{33} + (a^{+2}_{13}+a^{+2}_{23})s_{13}]\,\sigma'_3 \qquad (47b)$$

$$(\varepsilon_1-\varepsilon_2) + \tau_{\sigma_{(s_{11}-s_{12})}}(\dot\varepsilon_1-\dot\varepsilon_2) = (s_{11}-s_{12})(a^{+2}_{13}-a^{+2}_{23})\,\sigma'_3 + [(s_{11}-s_{12})-\delta_{(s_{11}-s_{12})}]\,\tau_{\sigma_{(s_{11}-s_{12})}}(a^{+2}_{13}-a^{+2}_{23})\,\dot\sigma'_3 \qquad (47c)$$

$$\varepsilon_4 + \tau_{\sigma_{s_{44}}}\dot\varepsilon_4 = a^+_{23}a^+_{33}s_{44}\sigma'_3 + (s_{44}-\delta_{s_{44}})\,\tau_{\sigma_{s_{44}}}a^+_{23}a^+_{33}\dot\sigma'_3 \qquad (47d)$$

$$\varepsilon_5 + \tau_{\sigma_{s_{44}}}\dot\varepsilon_5 = a^+_{13}a^+_{33}s_{44}\sigma'_3 + (s_{44}-\delta_{s_{44}})\,\tau_{\sigma_{s_{44}}}a^+_{13}a^+_{33}\dot\sigma'_3 \qquad (47e)$$

$$\frac{\varepsilon_6}{2} + \tau_{\sigma_{(s_{11}-s_{12})}}\frac{\dot\varepsilon_6}{2} = (s_{11}-s_{12})\,a^+_{13}a^+_{23}\sigma'_3 + [(s_{11}-s_{12})-\delta_{(s_{11}-s_{12})}]\,\tau_{\sigma_{(s_{11}-s_{12})}}\,a^+_{13}a^+_{23}\dot\sigma'_3 \qquad (47f)$$

There are no derivative terms in Equation 47a and 47b due to the fact that only Type I strains are involved. Furthermore, on taking into account these equations and after an arduous but simple treatment it can be

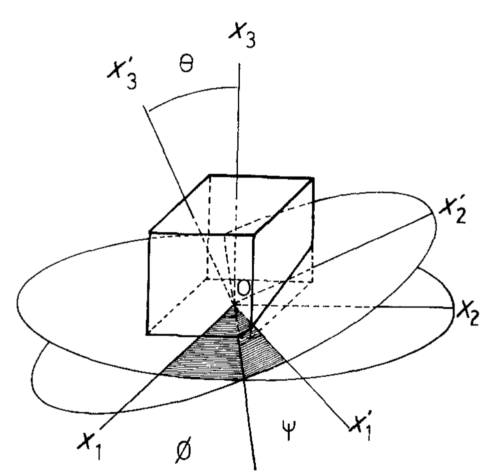

Figure 3 Hexagonal cell and Euler's angles.

shown that

$$\varepsilon_1 = \left|\left\{(a_{31}^2 s_{11} + a_{32}^2 s_{12} + a_{33}^2 s_{13}) + \left[(a_{31}^2 s_{11} + a_{32}^2 s_{12} + a_{33}^2 s_{13}) - \frac{(a_{31}^2 - a_{32}^2)}{2}\,\delta_{(s_{11}-s_{12})}\right]\omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}}\right\}\Big/(1 + \omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}}) - \frac{i\omega(a_{31}^2 - a_{32}^2)\,\delta_{(s_{11}-s_{12})}\tau_{\sigma_{s_{11}-s_{12}}}}{2(1 + \omega^2\tau^2_{\sigma_{s_{11}-s_{12}}})}\right|\sigma_3'$$

$$\varepsilon_2 = \left|\left\{(a_{31}^2 s_{12} + a_{32}^2 s_{11} + a_{33}^2 s_{13}) + \left[(a_{31}^2 s_{12} + a_{32}^2 s_{11} + a_{33}^2 s_{13}) + \frac{(a_{31}^2 + a_{32}^2)}{2}\,\delta_{(s_{11}-s_{12})}\right] \times \omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}}\right\}\Big/(1 + \omega^2\tau^2_{\sigma_{(s_{11}-s_{13})}} + \frac{i\omega(a_{31}^2 - a_{32}^2)\,\delta_{(s_{11}-s_{12})}\tau_{\sigma_{(s_{11}-s_{12})}}}{2(1 + \omega^2\tau^2_{\sigma_{s_{11}-s_{12}}})}\right|\sigma_3' \qquad (48)$$

$$\varepsilon_3 = [a_{33}^2 s_{33} + (1 - a_{33}^2)s_{13}]\,\sigma_3'$$

$$\varepsilon_4 = \left[\frac{s_{44} + (s_{44} - \delta_{s_{44}})\,\omega^2\tau_{\sigma_{s_{44}}} - i\omega\tau_{\sigma_{s_{44}}}\delta_{s_{44}}}{1 + \omega^2\tau^2_{\sigma_{s_{44}}}}\right] a_{33}a_{32}\sigma_3'$$

$$\varepsilon_5 = \left[\frac{s_{44} + (s_{44} - \delta_{s_{44}})\,\omega^2\tau_{\sigma_{s_{44}}} - i\omega\tau_{\sigma_{s_{44}}}\delta_{s_{44}}}{1 + \omega^2\tau^2_{\sigma_{s_{44}}}}\right] a_{33}a_{31}\sigma_3'$$

$$\varepsilon_6 = \left[\frac{(s_{11} - s_{12}) + (s_{11} - s_{12} - \delta_{(s_{11}-s_{12})})\,\omega^2\tau^2_{\sigma_{s_{11}-s_{12}}} - i\omega\tau_{\sigma_{(s_{11}-s_{12})}}\delta_{(s_{11}-s_{12})}}{1 + \omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}}}\right] 2a_{31}a_{32}\sigma_3'$$

On combining these equations with Equation 22 leads to

$$\varepsilon_1' = \left\{[s_{\mathrm{I}} + (s_{\mathrm{I}} - \delta s_{\mathrm{I}})\omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}} - i\omega\tau_{\sigma_{(s_{11}-s_{12})}}\delta s_{\mathrm{I}}]/(1 + \omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}}) - a_{13}^2 a_{33}^2 \times \frac{s_{44} + (s_{44} - \delta_{s_{44}})\omega^2\tau^2_{\sigma_{s_{44}}} - i\omega\tau_{\sigma_{s_{44}}}\delta_{s_{44}}}{(1 + \omega^2\tau^2_{\sigma_{s_{44}}})}\right\}\sigma_3' \qquad (49)$$

$$\varepsilon_3' = \left\{[s_{\mathrm{III}} + (s_{\mathrm{III}} - \delta s_{\mathrm{III}})\omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}} - i\omega\tau_{\sigma_{(s_{11}-s_{12})}}\delta s_{\mathrm{III}}]/(1 + \omega^2\tau^2_{\sigma_{(s_{11}-s_{12})}}) + a_{33}^2(1 - a_{33}^2) \times \frac{s_{44} + (s_{44} - \delta_{s_{44}})\omega^2\tau^2_{\sigma_{s_{44}}} - i\omega\tau_{\sigma_{s_{44}}}\delta_{s_{44}}}{(1 + \omega^2\tau^2_{\sigma_{s_{44}}})}\right\}\sigma_3'$$

where

$$s_{\mathrm{I}} = a_{13}^2 a_{33}^2 s_{11} + (a_{11}a_{32} - a_{12}a_{31})^2 s_{12} + [a_{13}^2(1 - a_{33}^2) + a_{33}^2(1 - a_{13}^2)]\,s_{13} + a_{13}^2 a_{33}^2 s_{33}$$

$$\delta s_{\mathrm{I}} = a_{13}^2 a_{33}^2 - (a_{11}a_{32} - a_{12}a_{31})^2\,\frac{\delta(s_{11} - s_{12})}{2}$$

$$s_{\mathrm{III}} = (1 - a_{33}^2)^2 s_{11} + 2a_{33}^2(1 - a_{33}^2)s_{13} + a_{33}^4 s_{33}$$

$$\delta s_{\mathrm{III}} = (1 - a_{33}^2)^2\,\frac{\delta(s_{11} - s_{12})}{2} \qquad (50)$$

In the particular case where $\{a\}$ is the orthogonal matrix of Euler's angles, that is,

$$\{a\} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \cos\psi\sin\phi + \cos\theta\cos\psi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \qquad (51)$$

and $\psi = \theta = 0$, then

$$\varepsilon_3' = s_{33}\sigma_3' \qquad (52)$$

and

$$\varepsilon_1' = s_{13}\sigma_3' \qquad (53)$$

Moreover

$$E^{-1} = \frac{\varepsilon_3'}{\sigma_3'} = s_{33}'(\omega\tau_\sigma) \qquad (54)$$

which is the anelastic generalization of the known equation of elasticity [5]. Since

$$s_{33}' = \sin^4\theta\, s_{11} + \sin^2\theta\cos^2\theta\,(2s_{13} + s_{44}) + \cos^4\theta\, s_{33} \qquad (55)$$

on taking into account Equations 52 and 53 leads to

$$E^{-1} = s_{33} \qquad (56)$$

and

$$v_{13} = v_{13}^{(1)} - iv_{13}^{(2)} = -\frac{\varepsilon_1'}{\varepsilon_3'} = -\frac{s_{13}}{s_{33}} \qquad (57)$$

which are invariant under a change of $\omega\tau_\sigma$. These results will be considered in a forthcoming paper, where the concepts developed will be applied to actual experimental data.

2.4. Hexagonal symmetry. Shear stress

It will be assumed that σ_{23}' is the shear stress applied on a cylindrical specimen. Then, as in the previous section, an integration over ψ is necessary to average correctly. Furthermore, as in the previous paragraph, it will be assumed that only single defects are present.

Under these conditions

$$G^{-1} = \frac{\int_0^{2\pi} s'_{44}\, d\psi}{2\pi} = \{s'_{IV} + [S'_{IV} - (1 - a_{33}^2)^2 \delta_{(s_{11}-s_{12})}] \times \omega^2 \tau^2_{\sigma_{(s_{11}-s_{12})}} - i\omega\tau_{\sigma_{(s_{11}-s_{12})}} (1 - a_{33}^2)^2\} / (1 + \omega^2 \tau^2_{\sigma_{(s_{11}-s_{12})}}) + \frac{(1 + a_{33}^2) - 2(1 - a_{33}^2)a_{33}^2}{2} \times [s_{44} + (s_{44} - \delta_{s_{44}})\, \omega^2 \tau^2_{\sigma_{s_{44}}} - i\omega\tau_{\sigma_{s_{44}}} \delta_{s_{44}}] / (1 + \omega^2 \tau^2_{\sigma_{s_{44}}}) \quad (58)$$

where

$$s'_{IV} = (1 - a_{33}^2)\, [(s_{11} - s_{12}) + 2a_{33}^2\, (s_{11} - 2s_{13} + s_{33})] \quad (59)$$

which is only a function of θ.

2.5. Anelastic lost energy

The elastic energy is given by

$$W = \tfrac{1}{2} c_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} \quad (60)$$

so that

$$dW = c_{ijkl}\, \varepsilon_{ij}\, d\varepsilon_{kl} \quad (61)$$

and when the strain is a maximum, the total elastic energy accumulated in a quarter of a cycle is

$$W = \frac{\omega}{4} \int_0^T c_{ijkl}\, \varepsilon_{ij}\, \dot{\varepsilon}_{kl}\, dt \quad (62)$$

The energy dissipated per cycle is given by

$$\Delta W = \int_0^T c_{ijkl}\, \varepsilon_{ij}\, d\varepsilon_{kl} \quad (63)$$

Moreover, under a uniaxial stress

$$\begin{aligned} \varepsilon_{ij} &= (s_{ij}^{R} - iS_{ij}^{I})\, \sigma_{01} \exp(i\omega t) \\ \dot{\varepsilon}_{kl} &= \omega\, (s_{kl}^{I} + is_{kl}^{R})\, \sigma_{01} \exp(i\omega t) \end{aligned} \quad (64)$$

where R and I denote the real and imaginary component, respectively. In terms of the stress tensor

$$W = \frac{\omega}{4} \int_0^{2\pi/\omega} \sigma_{kl}\, \dot{\varepsilon}_{kl}\, dt = \sigma_{kl} \frac{\pi}{4} \sigma_{01}\, s_{kl11}^{R} \quad (65)$$

and

$$\Delta W = \int_0^{2\pi/\omega} \sigma_{kl}\, \dot{\varepsilon}_{kl}\, dt = \sigma_{kl}\, \pi\, \sigma_{01}\, s_{kl11}^{I} \quad (66)$$

When Equations 65 and 66 are full developed, the following expressions are obtained

$$W = \frac{\sigma_{01}\pi}{4} [\sigma_{11} s_{1111}^{R} + \sigma_{22} s_{2211}^{R} + \sigma_{33} s_{3311}^{R} + 2(\sigma_{12} s_{1211}^{R} + \sigma_{13} s_{1311}^{R} + \sigma_{23} s_{2311}^{R})] \quad (67)$$

and

$$\Delta W = \sigma_{01}\pi\, [\sigma_{11} s_{1111}^{I} + \sigma_{22} s_{2211}^{I} + \sigma_{33} s_{3311}^{I} + 2(\sigma_{12} s_{1211}^{I} + \sigma_{13} s_{1311}^{I} + \sigma_{23} s_{2311}^{I})] \quad (68)$$

Each term of these equations describes the lost and stored energy raised by a particular strain mode. In a two index notation, which does not mean tensorial character, Equations 67 and 68 can be written as

$$W = W_{11} + W_{12} + W_{13} + W_{14} + W_{15} + W_{16} \quad (69)$$

$$\Delta W = \Delta W_{11} + \Delta W_{12} + \Delta W_{13} + \Delta W_{14} + \Delta W_{15} + \Delta W_{16} \quad (70)$$

Furthermore, the following quotients give the expressions for the internal friction due to each strain mode

$$\begin{aligned} \tan\phi_1 &= \frac{\Delta W_{11}}{4W_{11}} = \frac{s_{11}^{I}}{s_{11}^{R}} & \tan\phi_2 &= \frac{\Delta W_{12}}{4W_{12}} = \frac{s_{12}^{I}}{s_{12}^{R}} \\ \tan\phi_3 &= \frac{\Delta W_{3}}{4W_{13}} = \frac{s_{13}^{I}}{s_{13}^{R}} & \tan\phi_4 &= \frac{\Delta W_{14}}{4W_{14}} = \frac{s_{14}^{I}}{s_{14}^{R}} \\ \tan\phi_5 &= \frac{\Delta W_{15}}{4W_{15}} = \frac{s_{15}^{I}}{s_{15}^{R}} & \tan\phi_6 &= \frac{\Delta W_{16}}{4W_{16}} = \frac{s_{16}^{I}}{s_{16}^{R}} \end{aligned} \quad (71)$$

where the ϕ_i are the phase angles between the longitudinal stress and the corresponding strain. For example, ϕ_1 is the angle defined by Equation 8, in the more restricted unidimensional problem.

Finally, the results given by Equation 28 will be considered, as an example. In fact

$$\begin{aligned} \tan\phi_1 &= \frac{-\omega\tau_{\sigma_{(s_{11}-s_{12})}} \dfrac{2\,\delta_{(s_{11}-s_{12})}}{3}}{s_{11} + (s_{11} - \frac{2}{3}\delta_{(s_{11}-s_{12})})\, \omega^2 \tau^2_{\sigma_{(s_{11}-s_{12})}}} \\ \tan\phi_2 &= \frac{\omega\tau_{\sigma_{(s_{11}-s_{12})}} \dfrac{\delta_{(s_{11}-s_{12})}}{3}}{s_{12} + \left(s_{12} + \dfrac{\delta_{(s_{11}-s_{12})}}{3}\right) \omega^2 \tau^2_{\sigma_{(s_{11}-s_{12})}}} \end{aligned} \quad (72)$$

and Equation 28 can be written in the alternative form

$$\begin{aligned} \varepsilon_1 &= |\varepsilon_{01}| \exp(-i\phi_1) \exp(i\omega t) \\ \varepsilon_2 &= |\varepsilon_{02}| \exp(-i\phi_2) \exp(i\omega t) \end{aligned} \quad (73)$$

The corresponding modulus and phase diagram is shown in Fig. 4. It can be seen very easily that

$$\tan(\phi_2 - \phi_1) = \frac{\nu^{(2)}}{\nu^{(1)}} = -\frac{\omega\tau_{\sigma_{(s_{11}-s_{12})}} \dfrac{\delta_{(s_{11}-s_{12})}}{3} (s_{11} + 2s_{12})}{s_{11}s_{12} + \left(s_{11} - \dfrac{2\delta_{(s_{11}-s_{12})}}{3}\right)\left(s_{12} + \dfrac{\delta_{(s_{11}-s_{12})}}{3}\right) \omega^2 \tau^2_{\sigma_{(s_{11}-s_{12})}}} \quad (74)$$

is the phase lag between longitudinal and transversal strains. On using the same arguments as for Equations 30 and 31, the derivative

$$\frac{\partial\, (\nu^{(2)}/\nu^{(1)})}{\partial\, \omega\tau_\sigma} = 0$$

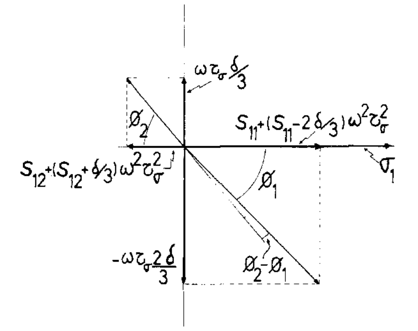

Figure 4 Phase diagram of longitudinal and transversal strains for the cubic symmetry. X_1' coincident with X_1.

leads to

$$\omega\tau_\sigma = \frac{s_{11}s_{12}}{\left(s_{11} - \frac{2\delta_{(s_{11}-s_{12})}}{3}\right)\left(s_{12} + \frac{\delta_{(s_{11}-s_{12})}}{3}\right)} \cong 1 \tag{75}$$

and, if

$$\omega\tau_\sigma \to 0 \qquad \frac{v^2}{v^{(1)}} \to 0$$
$$\omega\tau_\sigma \to \infty \qquad \frac{v^{(2)}}{v^{(1)}} \to 0 \tag{76}$$

This behaviour is shown qualitatively in Fig. 2 and some characteristic values, indicated from 1 to 6, are given in Table III.

TABLE III Characteristic values for the points indicated from 1 to 6 in Fig. 2

Position	Value
1	$\dfrac{s_{11}}{\sqrt{3}\left(s_{11} - \frac{2\delta_{(s_{11}-s_{12})}}{3}\right)}$
2	$\dfrac{s_{11}}{s_{11} - \frac{2\delta_{(s_{11}-s_{12})}}{3}}$
3	$\left[\dfrac{s_{11}s_{12}}{\left(s_{11} - \frac{2\delta_{(s_{11}-s_{12})}}{3}\right)\left(s_{12} + \frac{\delta_{(s_{11}-s_{12})}}{3}\right)}\right]^{1/2}$
4	$-\dfrac{s_{12} + \frac{\delta_{(s_{11}-s_{12})}}{3}}{s_{11} - \frac{2\delta_{(s_{11}-s_{12})}}{3}}$
5	$\left[\dfrac{s_{11}s_{12}}{\left(s_{11} - \frac{2\delta_{(s_{11}-s_{12})}}{3}\right)\left(s_{12} + \frac{\delta_{(s_{11}-s_{12})}}{3}\right)}\right]^{1/2} \dfrac{s_{11} + 2s_{12}}{2(s_{11} + s_{12})}$
6	$\dfrac{\delta_{(s_{11}-s_{12})}(s_{11} + 2s_{12})}{6s_{11}\left(s_{11} - \frac{2\delta_{(s_{11}-s_{12})}}{3}\right)}$

3. Discussion and conclusions

The most interesting aspects of the theory developed are those concerning the energetic interpretations given in paragraph 2.5. In fact, experimental values are usually found in the literature, where the lost energy is appreciable in longitudinal vibrations, while the modulus does not show the expected variation. This behaviour can be attributed to the loss produced by other strains, than the one in the direction of the applied stress, which are not measured in these experiments.

Due to the higher elastic anisotropy of Poisson's ratio in single crystals [9, 10], than both Young's and shear modulus, a similar behaviour should be expected for time-dependent events. This anisotropy is reflected in several equations presented in the paper, and the coupling between the two perpendicular strains on the anelastic behaviour of Poisson's ratio is evident. Therefore, a more substantial information can be obtained from measurements of Poisson's ratio than from those of Young's or shear moduli.

The proposed model might give also an explanation of the wide scatter in experimental values of Poisson's ratio. In fact, even if Young's or shear modulus do not show appreciable relaxation, Poisson's ratio can be affected by frequency or temperature due to anelastic effects.

A set of equations were given to calculate relaxation and lost energies as functions of direction. These can be used to interprete data in single crystals. Pole diagrams and appropriate distribution functions should be used for polycrystals, for a correct average.

Finally, the concepts developed will be applied to actual experimental data in forthcoming papers.

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